Distance-based consensus modeling in Multiple Criteria Group Decision Making *

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Outline

- AHP-based individual rankings
- Aggregation through Choquet integral
- An ideal coalition
- Minimum distance consensus
- Consensus measure and power

Basic information

- $A = \{a_1, ..., a_M\}$ set of alternatives
- $C = \{c_1, ..., c_N\}$ set of criteria
- $E = \{e_1, ..., e_K\}$ set of experts

$V^k = \left[v_{ij}^k\right]$ matrix of ordinal rankings of expert *k*, v_{ij}^k in [0,1]

How to build V^k

- The elements of the matrix can be obtained using one of the many pairwise comparison methods proposed in the literature. The choice of the most appropriate method based on, perhaps, the criterion that it generates more reliable outcomes. Here we assume to adopt the Saaty's AHP

- Accordingly, each column of V^k represents the unique eigenvector associated with the sole positive eigenvalue of the pairwise comparison matrix to the corresponding criterion (Saaty, 1980)

Aggregation of experts' valuations

- kth expert's overall score of each alternative is obtained through the row-aggregation of the elements of the matrix V^k
- We assume that the aggregation is carried out using the Choquet integral (Choquet, 1953; Grabisch and Labreuche, 2010)
- Capacity depends on each expert judgement

Choquet aggregation

$$C_{\mu^k}\left(v_{i1}^k,\ldots,v_{iN}^k\right)=\sum_{j=1}^{N} v_{ij}^k\left(\mu^k(A_{\pi_j})-\mu^k(A_{\pi_{j+1}})\right)$$

$$v_{i\pi_1}^k \leq \ldots \leq v_{i\pi_N}^k$$

$$A_{\pi_j} = \{\pi_j,\ldots,\pi_N\}$$

μ^k *capacity associated to* e_k

The overall score of alternatives

- Let denote with v_i^k the overall score of a_i with respect to e_k
- Accordingly, we obtain for each expert e_k a vector $v^k = \left(v_1^k, \ldots, v_M^k\right)$ of the overall score of alternatives
- How to determine the group ranking?

Group ranking via weighting methods

- Methods based on the comparison of weighted sums of the scores have been widely used and the problem of determining the weights was addressed at first by Marquis de Borda (1781) and later by Kendall (1962), Kemeny and Snell (1962), Blin (1976), Cook and Seiford (1982), Cook and Kress (1990), and others

- In methods like Borda-Kendall’s one, weights are determined a priori without any “negotiation” between experts

Consensual group rankings

- Achieving an agreement with respect to a set of weights compatible with the preferences of experts is possible through the introduction, e.g., of distance-based consensus

- Distance-based consensus models aiming at maximizing the consensus between experts have been introduced, among others, by Cook et al. (1996), Cook (2006), Contreras (2010)

Driving coalition

- We find the most important simple majority coalition and we denote, as before, with v_i^k the score of the alternative a_i with respect to expert e_k in the coalition

- We use the coalition as a driver for finding the overall group consensual ranking of the alternatives

- The rankings of the experts in the above coalition are aggregated through the Choquet integral

Ideal ranking

$$C_{\mu^*}\left(v_i^{k_1^*},\ldots,v_i^{k_L^*}\right)=\sum_{l=1}^{L} v_i^{k_l^*}\left(\mu^*(A_l)-\mu^*(A_{l+1})\right)=v_i^*$$

- Denote with $v^* = (v_1^*,\ldots,v_M^*)$ the alternatives' ranking in the most important coalition (ideal ranking)

Overall consensus

- Consensual ranking is determined though constrained minimization of a distance function $\delta(u, v)$
- The approach is static and deterministic
- The first step is to define a set of desirable properties/axioms the distance should satisfy

Some distances

$$\delta(u,v) = \left(\sum_{i=1}^{n} \left| u_i - v_i \right|^p \right)^{1/p} \quad p \geq 1, \qquad \textit{Minkowski}$$

$$\delta(u,v) = \max \left\{ \left| u_1 - v_1 \right|, \ldots, \left| u_n - v_n \right| \right\} \qquad \textit{Chebyshev}$$

$$\delta(u,v) = 1 - \frac{\sum_{i=1}^{n} u_i v_i}{(\sum_{1=1}^{n} u_i^2)^{1/2} (\sum_{i=1}^{n} v_i^2)^{1/2}} \qquad Co\sin e$$

Minimum distance consensus 1

- Denoting with $V = \{v^1, ..., v^K\}$ the set of vectors of experts' rankings and with v a vector in the set of all possible rankings, overall consensual ranking is obtained solving the following minimization problem

$$Min_v F(\delta(v^1, v), ..., \delta(v^K, v))$$

$$s.t. \ \varepsilon_L \leq \delta(v^*, v) \leq \varepsilon_R$$

where v^* denotes ideal ranking

Weighted mean of distances

- Which assumptions?

$$F(\delta,...,\delta)=\delta$$

$$F(\delta_1,...,\delta_{k-1},\delta_k,\delta_{k+1},...,\delta_K)\neq F(\delta_1,...,\delta_{k-1},\overline{\delta_k},\delta_{k+1},...,\delta_K)\ \ if\ \ \delta_k\neq\overline{\delta_k}$$

$$F(\delta_1,...,\delta_K)=f_1(\delta_1)\circ f_2(\delta_2)\circ...\circ f_K(\delta_K)$$

$$F(\alpha\delta_1,...,\alpha\delta_K)=\alpha F(\delta_1,...,\delta_K)\ \ \alpha>0$$

Minimum distance consensus 2

- For p=1, the minimum problem can be represented, assuming that the importance of expert e_k is w_k (normalized set of positive weights) as

$$\min_v \sum_{k=1}^{K} \sum_{i=1}^{M} w_k \left| v_i^k - v_i \right|$$

$$\varepsilon_L \leq \sum_{i=1}^{M} \left| v_i^* - v_i \right| \leq \varepsilon_R$$

Minimum distance consensus 3

- Change of variables

$$x_{ki} = \frac{1}{2}\left[\left|v_i^k - v_i\right| + (v_i^k - v_i)\right]$$

$$y_{ki} = \frac{1}{2}\left[\left|v_i^k - v_i\right| - (v_i^k - v_i)\right]$$

$$x_{*i} = \frac{1}{2}\left[\left|v_i^* - v_i\right| + (v_i^* - v_i)\right]$$

$$y_{*i} = \frac{1}{2}\left[\left|v_i^* - v_i\right| - (v_i^* - v_i)\right]$$

Minimum distance consensus 4

- LP in consensus reaching

$$\min \sum_{k=1}^{K} \sum_{i=1}^{M} w_k (x_{ki} + y_{ki})$$

$$\varepsilon_L \leq \sum_{i=1}^{M} (x_{*i} + y_{*i}) \leq \varepsilon_R$$

$$v_i + x_{*i} - y_{*i} = v_i^*$$

$$v_i + x_{ki} - y_{ki} = v_i^k$$

$$k = 1,...., K \; ; \; i = 1,..., M$$

Consensus measure 1

- A consensus measure is introduced to show how each expert contributes to consensus

$$Cons\,(V,E) : R(A)^K \times 2^{E_2} \to [0,1] \qquad V = (v^1,\ldots,v^K)$$

where $R(A)$ is the set of rankings over *A* and 2^{E_2} is the subset of the power set of *E* containing those elements whose cardinality is at least 2

Consensus measure 2

- Properties to be satisfied

  1. weak unanimity: maximum consensus is achieved when all preferences are the same

  2. anonymity: simmetry with respect to decision makers

  3. neutrality: simmetry with respect to alternatives

Consensus measure 3

- A consensus measure based on distance $\delta(u,v)$

$$Cons(V,P) = 1 - \frac{\sum_{e_h, e_k \in P, h<k} \delta(v^h, v^k)}{\frac{|P|!}{2(|P|!-2)} \max_{v_h, v_k} \delta(v^h, v^k)}$$

$$P \in 2^{E_2}$$

Consensus power

- The marginal contribution to consensus of expert e_k with respect to V can be determined according to the following index

$$\gamma_k(V) = \frac{\sum_{J \in X(k)} \left(Cons(V, J \cup \{e_k\}) - Cons(V, J) \right)}{|X(k)|}$$

$$X(k) = \left\{ J \in 2^{E_2} : e_k \notin J \right\}$$

Consensus contribution ranking

- Using these indexes, we introduce a priority ordering of the experts valuating their contribution to consensus, taking care of their importance

$$\gamma_k^{'}(V) = w_k \gamma_k(V) - \min\{w_1 \gamma_1(V), \ldots, w_K \gamma_K(V)\}$$

$$\rho_k(V) = \frac{\gamma_k^{'}(V)}{\sum_{k=1}^{K} \gamma_k^{'}(V)} \quad when \quad \sum_{k=1}^{K} \gamma_k^{'}(V) \neq 0 \qquad otherwise \quad is \quad \frac{1}{K}$$

$$k = 1, \ldots, K$$

Future work

- Study of the impact experts' opinions changing on consensual dynamics
- Consistency evaluation
- Consensus power and minimum winning coalitions
- Extension to vague environments (fuzzy preferences)

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